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J. Phys. A: Math. Gen. 18 (1985) 1229-1238. Printed in Great Britain

A criterion for disorder solutions of spin models

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Received 16 October 1984

Abstract. A simple criterion is given which provides disorder solutions for spin models of the Ising or Potts type. New disorder solutions are thus obtained, in particular for the Potts model on a Kagomé lattice and for the general anisotropic Ising model on a three-dimensional cubic lattice. The validity of the disorder solution, when extended outside the physical domain, is also discussed.

1. Introduction

A great variety of anisotropic models of the Ising or Potts type (with different coupling constants in the different directions) are known to possess remarkable submanifolds in the space of parameters, where the partition function is computable and takes a very simple form (Enting 1977). These so-called disorder solutions provide a precious insight into the behaviour of the partition function in its anisotropic parameters. Different techniques have been used to obtain these solutions for the various models: methods related to crystal growth (Enting 1978, Welberry and Miller 1978), to Markov processes (Verhagen 1976, Rujan 1982, 1984), or else to transfer matrices (Baxter 1985). In most cases, the problem of computing the partition function was compared with an equivalent one in another field, where appropriate techniques for reaching the solution were available. However, such a change in the point of view was in fact unnecessary, as will be shown in the following.

Indeed, all these methods rely on the same simple mechanism: a certain local decoupling of the spin degrees of freedom which results in an effective reduction of dimensionality for the spin system. Such a property is provided by a local condition, bearing on the Boltzmann weight of the elementary cell generating the lattice. It is this criterion that we shall define and study in § 2 of this paper. Hence, new disorder solutions will be produced (e.g. for the Potts model on a Kagomé lattice), together with three-dimensional generalisations (Ising models on a cubic lattice).

It will appear that particular boundary conditions must be imposed in order to easily obtain the disorder solutions for the partition functions. *A priori*, when coming back to standard periodic boundary conditions, this should limit these solutions to the physical domain only. In § 3, we shall study, using the simple example of a strip

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model, the meaning of the disorder solution, when extended to a larger domain. It will be seen that it is still related, through analytic continuation, to the usual partition function with periodic boundary conditions.

2. Local criterion for disorder solutions

In this section, we shall exhibit a local criterion to impose on the Boltzmann weight of the elementary cell, and show how to obtain a resulting disorder solution for the partition function. The procedure will be carried out for several models successively, including already known ones, like the anisotropic triangular Ising model with a field, or the checkerboard Potts model, but also new ones, such as the Potts model on a Kagomé lattice, and the general anisotropic cubic Ising model.

2.1. Triangular Ising model with a magnetic field

In the following, the different lattices will be considered as staggered lattices with various elementary generating cells. Figure 1 summarises the notations in the case of the triangular lattice, for which the Boltzmann weight of the elementary cell will be

$$w_n(\sigma_i, \sigma_j, \sigma_k) = \exp(-K_1\sigma_i\sigma_j - K_2\sigma_j\sigma_k - K_3\sigma_k\sigma_i - H_1\sigma_i - H_2\sigma_j - H_3\sigma_k),$$
$$H = H_1 + H_2 + H_3 \qquad (\sigma_{i,j,k} \in \mathbb{Z}_2).$$

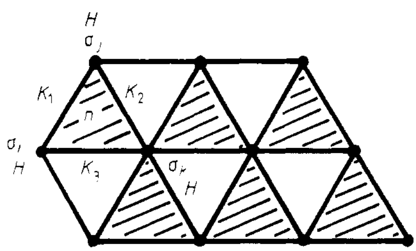

Figure 1. Ising model with a magnetic field on a triangular lattice.

In every case, the criterion will be defined by the following condition: after summation over some of its spins (to be defined in each case), the Boltzmann weight associated with the elementary cell must not depend on the remaining spins any longer. For instance, for the triangular lattice, we shall require that

$$\sum_{\sigma_j} w_n(\sigma_i, \sigma_j, \sigma_k) = \lambda(K_1, K_2, K_3, H)$$

is independent of σ_i and σ_k.

For an anisotropic Ising model with a magnetic field, with nearest neighbour interactions only, this leads to the following subvariety, in the space of parameters:

$$t_1t_2t_3(1+t_1)^2(1+t_2)^2(1-t_3)^2(1-z)^2 + 4(1+t_1t_2t_3)(t_1+t_2t_3)(t_2+t_1t_3)(t_3+t_1t_2)z = 0,$$
$$t_i = \tanh K_i \qquad (i = 1, 2, 3), \qquad z = e^{2H}, \tag{1}$$

in which one recognises a generalisation of Verhagen's condition (1976).

Let us now impose particular boundary conditions for the lattice: on the upper layer, all K_3 interactions are missing, so that the spins of the upper layer only interact with those below. It immediately follows that if one sums over all the spins of the upper layer and if one requires the disorder condition (1), the same boundary conditions reappear for the next layer. On the other hand, the partition function per site Z can be defined by

$$Z^{dN} = \sum_{\{\sigma\}} \prod_{n=1}^{N} w_n(\sigma) \tag{2}$$

where σ stands for all the different spins, n denotes the different cells, of total number N, and d the number of sites per cell. (This definition will be valid for all the different cases studied in this section; $d = 1$ for the triangular lattice.) Iterating the procedure allows one to perform the summation over σ in the partition function per site (2) and leads one to an exact expression for the latter $Z_{\rm D}$, when restricted to subvariety (1):

$$Z_{\rm D} = \lambda, \qquad \lambda = \frac{2(1-t_3^2)^{1/2}}{(1-t_1^2)^{1/2}(1-t_2^2)^{1/2}} \left(-\frac{t_1 t_2}{t_3}\right)^{1/2}. \tag{3}$$

This also gives Verhagen's solution (1976) as a particular case.

2.2. *Checkerboard Potts model*

The checkerboard lattice can be seen as generated by an elementary square cell (see figure 2). The associated elementary Boltzmann weight will be defined by

$$w_n(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = a^{\delta_{\sigma_i\sigma_j}} b^{\delta_{\sigma_j\sigma_k}} c^{\delta_{\sigma_k\sigma_l}} d^{\delta_{\sigma_l\sigma_i}} \qquad (\sigma_{i,j,k,l} \in \mathbb{Z}_q).$$

The corresponding criterion

$$\sum_{\sigma_j, \sigma_k} w_n(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = \lambda(a, b, c, d)$$

provides $q^2 - 1$ conditions, which in fact degenerate into the unique equation

$$\frac{1/d - 1}{1/d + q - 1} = \frac{a-1}{a+q-1}\,\frac{b-1}{b+q-1}\,\frac{c-1}{c+q-1}. \tag{4}$$

An iterative procedure similar to the one previously developed, together with analogous boundary conditions (an open lattice with no interactions d on the upper layer), reveals the exact analytical expression of the partition function per site $Z_{\rm D}$ ((2) with $d = 2$), on the variety (4):

$$Z_{\rm D} = \lambda^{1/2}, \qquad \lambda = \frac{(a+q-1)(b+q-1)(c+q-1)}{1/d + q - 1}. \tag{5}$$

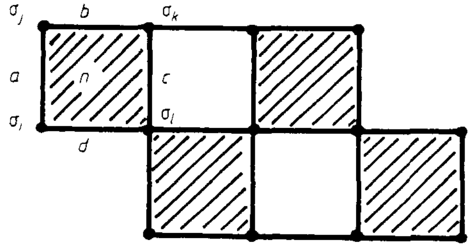

Figure 2. Potts model on a checkerboard lattice.

This is precisely the result which had been suggested by diagrammatic expansions (Jaekel and Maillard 1984), and proven by means of a transfer matrix method (Baxter 1985).

Let us remark that the criterion used by Baxter is very similar to the one introduced here. Indeed, let us consider the checkerboard lattice as an IRF (interaction round a face) model, generated by the elementary square cell of figure 3, with the following associated Boltzmann weight:

$$\bar{w}(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = \sum_s a^{\delta_{s\sigma_j}} b^{\delta_{s\sigma_i}} c^{\delta_{s\sigma_l}} d^{\delta_{s\sigma_k}} e^{\delta_{\sigma_i\sigma_l}} e^{-\delta_{\sigma_k\sigma_l}}$$

where e is an irrelevant coupling constant. Then Baxter's criterion is just that

$$\sum_{\sigma_j} \bar{w}(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = \lambda(a, b, c, d)$$

is independent of σ_i, σ_k, σ_l. The same kind of iterating procedure as the previous one leads to the results (4) and (5). Moreover, when one restricts this criterion to the Ising case ($q = 2$), one also recovers Enting's condition (1977) for a disorder solution of the checkerboard Ising model, which was obtained by means of a 'conditional probability' associated with an elementary cell:

$$P(\sigma_j | \sigma_i, \sigma_k, \sigma_l) = \lambda(1 + \sigma_j p(\sigma_i, \sigma_k, \sigma_l)).$$

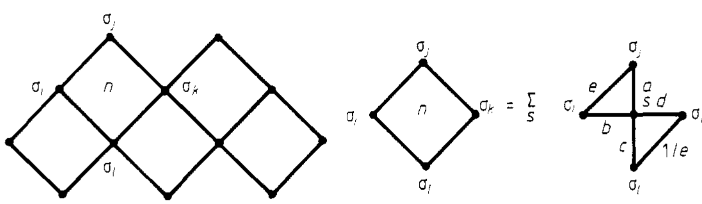

Figure 3. Potts model seen as an IRF model.

Let us finally remark that, although identical for the checkerboard Potts model, the two criteria developed here have in general different realms of application and are not equivalent.

2.3. *Potts model on a Kagomé lattice*

The Kagomé lattice can be generated by the elementary cell represented by figure 4, with its associated Boltzmann weight

$$w_n(\sigma_i, \sigma_j, \sigma_k, \sigma_l, \sigma_m) = a^{\delta_{\sigma_i\sigma_m}} b^{\delta_{\sigma_m\sigma_l}} c^{\delta_{\sigma_l\sigma_i}} a^{\delta_{\sigma_m\sigma_k}} b^{\delta_{\sigma_m\sigma_j}} c^{\delta_{\sigma_j\sigma_k}} \qquad (\sigma_{i,j,k,l,m} \in \mathbb{Z}_q).$$

The corresponding version of the criterion can be written as

$$\sum_{\sigma_j\sigma_k\sigma_m} w_n(\sigma_i, \sigma_j, \sigma_k, \sigma_l, \sigma_m) = \lambda'(a, b, c)$$

and is independent of σ_i and σ_l. It leads to the following unique condition:

$$\frac{1/c - 1}{1/c + q - 1} = \frac{a-1}{a+q-1}\,\frac{b-1}{b+q-1} \tag{6}$$

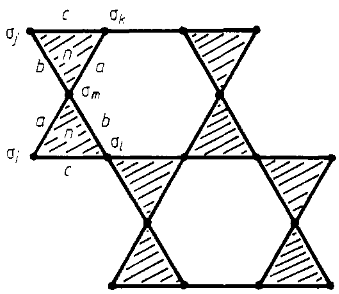

Figure 4. Potts model on a Kagomé lattice.

with the partition function per site then taking the following rational expression (take $d = 3$ in (2)):

$$Z_D = \lambda^{1/3}, \qquad \lambda = q\frac{(a+q-1)^2(b+q-1)^2}{(1/c+q-1)^2}. \tag{7}$$

One will remark that, by construction, the disorder variety appears to be the same as that for the triangular lattice, and moreover that the respective expressions for the partition functions per site are similar (compare (7) with (5)). This new exact particular solution for the Potts model on a Kagomé lattice is found to be in agreement with its special case of the Ising model ($q = 2$). Indeed, the partition function of the latter is exactly known for any values of the parameters, and can be expressed in the form of elliptic integrals (Green and Hurst 1964). One easily checks that for the relation (6) between the parameters, the modulus of the elliptic integrals vanishes, providing the required rational expression (7).

2.4. General cubic Ising model

The previously introduced criterion provides a quick and easy derivation of disorder solutions. Moreover this holds for a large class of two-dimensional models. Another advantage lies in the immediate generalisation to three dimensions.

Let us consider the most general anisotropic cubic Ising model, with nearest neighbour interactions only, that can be built in a staggered way starting from the elementary cubic cell shown by figure 5. It will depend on 12 parameters, i.e. on

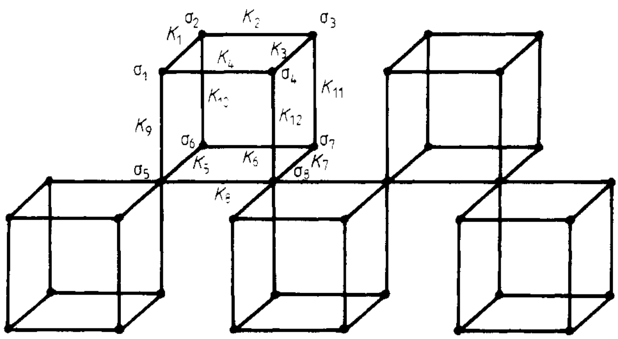

Figure 5. General cubic Ising model.

the different coupling constants associated with the 12 links of the elementary cube generating the lattice. Let us denote its Boltzmann weight by

$$w_n(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8)$$
$$= \exp(K_1\sigma_1\sigma_2 + K_2\sigma_2\sigma_3 + K_3\sigma_3\sigma_4 + K_4\sigma_4\sigma_1)$$
$$\times \exp(K_5\sigma_5\sigma_6 + K_6\sigma_6\sigma_7 + K_7\sigma_7\sigma_8 + K_8\sigma_8\sigma_5 + K_9\sigma_1\sigma_5$$
$$+ K_{10}\sigma_2\sigma_6 + K_{11}\sigma_3\sigma_7 + K_{12}\sigma_4\sigma_8) \qquad (\sigma_{i,i=1,8} \in \mathbb{Z}_2).$$

A three-dimensional generalisation of the disorder criterion is provided by

$$\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} w_n(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8) = \lambda(K_{i,i=1,12}) \tag{8}$$

which is independent of σ_5, σ_6, σ_7, σ_8, from which one must extract the set of equations between the coupling constants, representing the conditions for a disorder solution. Instead, and for reasons that will become clearer in the following, we shall introduce the following sequence of operations, as illustrated by figure 6. First, after summation over spins σ_2 and σ_4 we replace interactions K_1, K_2, K_{10} (resp. K_3, K_4, K_{12}) by equivalent ones L_1, L_2, L_{10}(resp. L_3, L_4, L_{12}), using a well known star-triangle transformation (Domb and Green 1972):

$$\exp(4L_i) = \frac{\cosh(K_i + K_j + K_k)\cosh(K_i - K_j - K_k)}{\cosh(K_i + K_j - K_k)\cosh(K_i - K_j + K_k)} \text{ and permutations of } (ijk),$$
$$i, j, k = 1, 2, 10 \qquad (\text{resp. } 3, 4, 12).$$

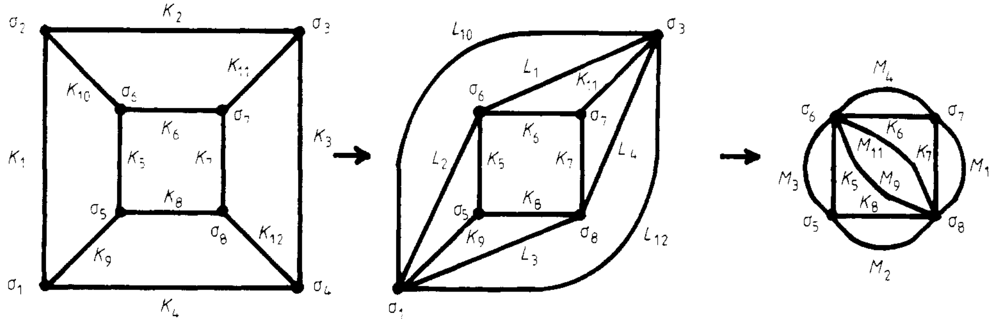

Figure 6. Star-triangle transformations in the cubic Ising model.

Then imposing the constraint

$$L_{10} + L_{12} = 0 \tag{9}$$

we can, after summing over the remaining two spins, make use of another star-triangle transformation: L_1, L_4 and L_{11} (resp. L_2, L_3 and L_9) become M_1, M_4 and M_{11} (resp. M_2, M_3 and M_9):

$$\exp(4M_i) = \frac{\cosh(L_i + L_j + L_k)\cosh(L_i - L_j - L_k)}{\cosh(L_i + L_j - L_k)\cosh(L_i - L_j + L_k)} \text{ and permutations of } (ijk),$$
$$i, j, k = 1, 4, 11 \qquad (\text{resp. } 2, 3, 9) \qquad (L_9 = K_9;\ L_{11} = K_{11}).$$

Finally, the disorder criterion for the cube is satisfied as soon as the following additional equations hold:

$$M_1+K_7=0, \qquad M_2+K_8=0, \qquad M_3+K_5=0,$$
$$M_4+K_6=0, \qquad M_9+M_{11}=0. \tag{10}$$

Collecting the equations in (9) and (10) provides a subvariety of codimension six in the 12-dimensional space of parameters, for which the disorder condition (8) is satisfied. In an iterative way completely similar to the two-dimensional one, the disorder criterion (8) can be used on the lattice, once adapted boundary conditions are chosen: an open lattice with the interactions K_5, K_6, K_7 and K_8 missing in the upper layer. Consequently, the partition function per site, when restricted to subvariety (9), (10) (take $d=4$ in (2)), becomes

$$Z_{\mathrm{D}}=\lambda^{1/4},$$
$$\lambda=\bar{\lambda}(K_1, K_2, K_{10})\bar{\lambda}(K_3, K_4, K_{12})\bar{\lambda}(L_1, L_4, K_{11})\bar{\lambda}(L_2, L_3, K_9),$$
$$\bar{\lambda}(K_i, K_j, K_k)=2[\cosh(K_i+K_j+K_k)\cosh(K_i+K_j-K_k)$$
$$\times\cosh(K_i-K_j+K_k)\cosh(K_i-K_j-K_k)]^{1/4}.$$

One will note that we could have performed the same sequence of transformations, but starting with spins σ_1 and σ_3, and ending with spins σ_2 and σ_4. A similar result would have followed.

Conditions (9), (10) obviously are a sufficient set of relations for satisfying (8). In general, the latter will provide a subvariety of codimension seven (2^3-1 homogeneous conditions) in the 12-dimensional space of parameters, where the partition function is known. But degeneracies can occur between these conditions, and lead to subvarieties of lower codimension. Such is precisely the case of the previous example (9), (10), thanks to the use of the star-triangle transformations. Another efficient way to decrease the codimensionality is to impose symmetries on the interactions, which remain compatible with the disorder conditions (8) or (9), (10). For instance, if one identifies the coupling constants K_1, K_2, K_3, K_4 respectively with K, K, K, $-K$, and K_5, K_6, K_7, K_8 respectively with K', K', K', $-K'$, and K_9, K_{10}, K_{11}, K_{12} with L, then (9) is automatically satisfied and the criterion (10) provides a subvariety of codimension one, in the three-dimensional space of parameters (K, K', L) defined by the equation

$$T'+\tau^2T=0, \qquad T=\tanh 2K, \qquad T'=\tanh 2K', \qquad \tau=\tanh L,$$

where the partition function is found to take the simple expression

$$Z_{\mathrm{D}}=\lambda^{1/4}, \qquad \lambda=2^4\frac{1}{(1-\tau^2)^2}\frac{(1-T'^2)^{1/2}}{(1-T^2)^{1/2}}$$

(Jaekel and Maillard 1985).

3. Analytical extension

The partition functions per site which have been computed in § 2 correspond to lattices with unusual boundary conditions. In the physical domain, where the coupling constants are real, these do not affect the partition function per site in the thermodynamic limit (this can be seen by using the Perron–Frobenius theorem for instance), and the expressions computed in § 2 also correspond to partition functions per site with standard periodic boundary conditions. Such an identification is also confirmed by expansions (Enting 1977). However, there are cases for which the disorder variety lies partially or even entirely outside the physical domain, where the boundary conditions are known to play an important role (Baxter 1982b). One can then legitimately wonder whether the expressions computed here have any relationship with the partition functions with standard boundary conditions. In order to shed some light on this problem, it seems better to exhibit an example where both expressions are exactly known. That is why we shall now study a simple illustrative model: the triangular Ising model with a magnetic field, on a strip.

The underlying lattice will be on a torus, with doubly periodic (transverse and longitudinal) boundary conditions, as represented by figure 7. The partition function per site Z will be defined by

$$Z^{2N} = \sum_{\{\sigma,\sigma'\}} \prod_{n=1}^{N} \exp[K(\sigma_n\sigma_{n+1} + \sigma'_n\sigma'_{n+1} + \sigma'_n\sigma_{n+1} + \sigma_n\sigma'_{n+1} + 2\sigma_{n+1}\sigma'_{n+1}) + H(\sigma_{n+1} + \sigma'_{n+1})].$$

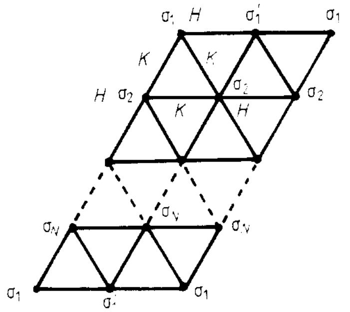

Figure 7. Ising model with a magnetic field on a triangular strip.

The latter is easily computed by means of the transfer matrix method and is given by the largest root of the following characteristic polynomial:

$$\Lambda^3 - [1 - 2(1+u)v]\Lambda^2 + [2u^2v + (1 + 2u - u^2)v^2]\Lambda - 2u^2(1+u)v^3 = 0 \tag{11}$$

where

$$u = e^{-4K}, \qquad v = \frac{1 - e^{-4K}}{(e^H + e^{-H})^2}, \qquad Z = e^{3K}(e^H + e^{-H})\Lambda^{1/2}.$$

The criterion for disorder, in its version of § 2.1 ((1), in the isotropic limit $K_1 = K_2 = K_3 = K$), also allows one to compute the partition function per site of the transversally open lattice, on a disorder variety:

$$Z_D = (e^{-2K} - e^{2K})^{1/2} \qquad (\Lambda_D = u^2), \tag{12}$$

when

$$e^{4K} + e^{2H} + e^{-2H} + 1 = 0 \qquad (v + u = 0).$$

As is easily seen on the corresponding limit of the characteristic polynomial (11):

$$[\Lambda - u^2][\Lambda^2 - (1+u)^2\Lambda - 2u^3(1+u)] = 0$$

the disorder solution Λ_D is also a root of the latter. Yet, one easily verifies that it does not identify with the largest root:

$$\Lambda = \frac{(1+u)^2}{2}\left\{1 + \left[1 + \left(\frac{2u}{1+u}\right)^3\right]^{1/2}\right\} \tag{13}$$

and thus that it does not give the partition function per site of the lattice with standard periodic boundary conditions. However, the fact that both expressions are roots of the same polynomial reveals the existence of an analytic continuation between them.

This simple example already indicates how careful one must be when using the disorder solution, in order to get additional information on the behaviour of the partition function per site, outside the physical domain. For instance, Bessis *et al* (1976) have shown, with the help of Lee-Yang's theorem (1952), that the partition function per site has an expansion of the form

$$\ln Z = -\tfrac{1}{2}\ln\left(\frac{u^{3/2}}{1-u}v\right) - \sum_{l\geq 1}\frac{v^l}{l}P_l(u) \tag{14}$$

where P_l is a polynomial in u. In the previous example

$$P_1(u) = 1 + u + u^2.$$

In these variables, which are convenient for applying Lee-Yang's theorem, the disorder condition reads $v + u = 0$ and one gets the impression that the expansion does not agree with the disorder solution (12) for the partition function. However, the validity domains of these expressions being different, one must replace the disorder solution by its analytic continuation (13), which does agree with the expansion (14).

4. Conclusion

The criterion developed here, under various forms, shows that disorder solutions for the partition functions of spin models can be simply and directly obtained, without necessarily borrowing techniques from other fields. The method is also easily adapted to the case of vertex models, where it gives a particular version for finding disorder points (Baxter 1982a). Moreover, the correspondence between vertex models and one-dimensional quantum spin chains of the *XYZ* field type (Peschel and Rys 1982, Peschel and Emery 1981) allows one, when translating the criterion, to find the energy of the ground state. These exact solutions are different from the ones obtained in the cases of complete integrability, and hence provide complementary information on the behaviour of the quantum spin chain with respect to its coupling constants.

The iteration procedure which leads to the disorder solution is very similar to the one used for deriving the inversion relation (Jaekel and Maillard 1984), and both happen to hold simultaneously for many models. In that case, the inversion and the spatial symmetries of the model generate an infinite group, which plays the role of an automorphy group for the partition function, and which can be used to provide, from the disorder variety, an infinity of transformed varieties where the partition function is known.

Finally, the rational expression that appears on the disorder variety also implies constraints on the analytical behaviour of the partition function, especially at the intersection with the critical varieties. The disorder solutions can thus be very useful for clarifying the phase diagrams of anisotropic models, which tend to be rather complex (ANNNI models (Peschel and Emery 1981), Ising model with a field (Lin and Wu 1979)).

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